ISRAEL JOURNAL OF MATHEMATICS, Vol. 64, No. 2, 1988

ON THE GEOMETRY OF P-CONVEX SETS FOR OPERATORS OF REAL PRINCIPAL TYPE

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ABSTRACT

A geometric criterion of P-convexity for supports is provided for sets whose boundary does not contain intervals of straight lines.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open bounded set and $P: \mathscr{D}'(\Omega) \rightarrow \mathscr{D}'(\Omega)$ be a linear differential operator with constant coefficients. The set Ω is called P-convex (with respect to supports) if for every set K_0 compact in Ω there exists a set K_1 compact in Ω, such that for every $u \in \mathscr{E}'(\Omega)$, $\operatorname{supp} P(-D)u \subset K_0$ implies $\operatorname{supp} u \subset K_1$. Notation $\mathscr{E}'(\Omega)$ is used here for the space of distributions with compact support.

A survey on P-convexity can be found in [2], Chapter X. We know only a few cases when a geometric criterion for P-convexity is established. They include: (1) $n = 2$, (2) $P = (\alpha, D)$ or P is elliptic, (3) P is the wave operator, (4) $n = 3$, $\partial\Omega \subset C^2$ and P has simple characteristics (see [2], [3], [4]).

This paper provides a criterion for P-convexity when P is an operator of real principal type and $\partial\Omega$ does not contain intervals of straight lines. There are no smoothness conditions for $\partial\Omega$.

In Section 2 of this paper we state the main theorem and prove the necessity. The proof of necessity is rather standard and is based on [1], [2]. In Section 3 we prove sufficiency. In Section 4 we give an example of a P-convex set and

[†] Research partially done at the Weizmann Institute of Science.

Received December 22, 1987 and in revised form September 23, 1988

discuss the relation between P-convexity and uniqueness in the Cauchy problem.

2. Main result. Necessity

Let $p(\xi)$ be the principal symbol of P. We suppose that p is real and $p(\xi)=0$ implies $p'(\xi)\neq 0$ when $\xi\neq 0$.

THEOREM 1. *Assume that $\Omega\subset\mathbf{R}^n$ is an open set and*

(1) *for every $x, y\in\partial\Omega$ there is α, $0<\alpha<1$, such that $\alpha x+(1-\alpha)y\notin\partial\Omega$.*

An open bounded set Ω satisfying (1) *is P-convex if and only if:*
There is no closed convex set F and an open convex set W, such that

(2)
(a) $F\subset\overline{\Omega}$,
(b) *$\partial F\cap W$ is an analytic hypersurface with normals lying in $p^{-1}(0)$,*
(c) *$\partial F\cap\partial\Omega$ is non-empty and compact in W.*

In order to prove necessity we wish first to examine convex characteristic surfaces.

LEMMA 1. *Let $K_1\subset K\subset\mathbf{R}^n$ be open convex sets and let σ_1 be a supporting function of K_1. Assume that*

(3) $K_2=\{x\in\mathbf{R}^n:(x,\xi)<\sigma_1(\xi),\ \xi\in p^{-1}(0)\setminus 0\}$.

If $u=0$ in K_1 and $Pu=0$ in K, then

(4) $u=0$ *in* $K_2\cap K$.

Lemma 1 is an elementary reformulation of Theorem 8.6.8 (implication (ii)$\Rightarrow$(i)), [2].

In what follows we will denote an open ball of radius $\rho>0$ centered at z by $B(z,\rho)$, and consider the following open convex set:

(5) $B_P(z,\rho)=\{x\in\mathbf{R}^n:(x,\xi)<(z,\xi)+\rho|\xi|,\ \xi\in p^{-1}(0)\setminus 0\}$.

LEMMA 2. *Let x_0 be a characteristic point of $\partial B(z,\rho)$, i.e., $p(x-z)=0$. There is a neighbourhood W of x_0 such that $\partial B_P(z,\rho)\cap W$ is an analytic hypersurface containing x_0. Moreover, if $\nu(x)$ is a normal to a plane supporting $B_P(z,\rho)$ at $x\in\partial B_P(z,\rho)\cap W$, then*

(6) *the face of $B_P(z,\rho)$ containing x is an interval of the bicharacteristic line $t\to x+tp'(\nu(x))$,*

(7) $\nu(x) \in p^{-1}(0)$.

We did not find a proof for Lemma 2 in literature, although it is elementary. We provide a proof here to keep the paper self-contained.

PROOF. 1. Consider the following set:

(8) $\Gamma_\varepsilon = \{x + tp'(x-z) : |t| < \varepsilon,\ |x - x_0| < \varepsilon,\ x \in \partial B(x,\rho),\ p(x-z) = 0\}$, $\varepsilon > 0$.

Since P is of real principal type, by the implicit function theorem equations

(9) $|x - z|^2 = \rho^2,\ p(x-z) = 0$

define near $x = x_0$ an analytic manifold γ of codimension 2. The normals to this manifold, by (9), lie in the span of $x - z$ and $p'(x-z)$. Therefore the map

(10) $(x,t) \to x + tp'(x-z),\ x \in \gamma,\ t \in \mathbf{R}$

is a smooth flow for a neighbourhood of x_0 when t is small and its image is an analytic hypersurface. Therefore, there exist $\varepsilon > 0$, such that Γ_ε is an analytic manifold of codimension 1 containing x_0.

2. Let us prove that there is a neighbourhood W of x_0, such that

(11) $\Gamma_\varepsilon = \partial B_P(x,\rho)$ in W.

First observe that for $\xi \in p^{-1}(0)$, $|\xi| = 1$, $x \in \gamma$ and $|t|$ small

(12) $(x - z + tp'(x-z), \xi) \leqq \rho$ with an equality only at $x - z = \rho\xi$.

Indeed, $x = z + \rho\xi$ is a point of a local maximum for any fixed ξ, $|t| < \varepsilon$:

$$d_x(x - z + tp'(x-z), \xi) = 0 \qquad \text{on } T_{z+\rho\xi}\gamma, \quad |t| < \varepsilon,$$

and

$$d_x^2(x - z + tp'(x-z), \xi) < \tfrac{1}{2} d_x^2(x - z, \xi) = -\frac{1}{2\rho} I < 0$$
$$\text{on } T_{z+\rho\xi}\gamma \setminus 0, \quad \text{for } |t| \text{ small.}$$

Therefore (12) is correct if $|x - z - \rho\xi| < \delta$ for some $\delta > 0$.
If $|x - z - \rho\xi| \geqq \delta$, then

$$(x - z, \xi) + t(p'(x-z), \xi) \leqq \rho - \delta^2/2\rho + O(|t|) < \rho$$

for $|t|$ small, and (12) follows.
By (12)

(13) $\Gamma_\varepsilon \subset \partial B_P(z,\rho)$.

It remains to show now that in a neighbourhood of x_0,

(14) $\partial B_P(z,\rho)\setminus\Gamma_\varepsilon = \varnothing$.

Suppose (14) is wrong and $y \in \partial B_P(z,\rho)\setminus\Gamma_\varepsilon$, $|y - x_0|$ is small. Since Γ_ε is smooth, there is $\lambda = 0$, $x \in \gamma$ and t, $|t| < \varepsilon$, such that

(15) $y = x + tp'(x - z) + \lambda(x - z)$.

Repeating the argument used for (12), one has

(16) $\max\limits_{\substack{\xi:|\xi|=1,\\ p(\xi)=0}} (y - z, \xi) = \rho + \lambda, \quad \lambda \neq 0$

and thus $y \notin \partial B_P(z,\rho)$, which contradicts (14). Thus (11) is proved.

3. Let $x + \partial B_P(x, p) \cap W$. Since (10) is a smooth flow, there is $x_1 \in \gamma$ and t, $|t| < \varepsilon$, such that

$$x = x_1 + tp'(x_1 - z).$$

Since $x_1 \in \gamma$, the vector $x_1 - z$ is characteristic. Therefore $x_1 - z$ is orthogonal to $p'(x_1 - z)$ by the Euler homogeneity theorem, and thus $x_1 - z$ is the normal to Γ_ε at x_1. Since $B_P(z,\rho)$ is convex and the plane $(x_1 - z)^\perp$ attached at x_1 contains $\{x + tp'(x_1 - z)\}_{t\in\mathbf{R}}$, the vector $x_1 - z$ is also the normal at x. Thus, since the normals $x_1 - z$ are different for different $x_1 \in \gamma$, the face of $B_P(z,\rho)$ supported by $x_1 - z$ is an interval of $x + tp'(x_1 - z)$. The relation (6) is proved. The relation (7) follows, since $v(x) = x_1 - z$, $x_1 \in \gamma$. Lemma 2 is proved.

PROOF OF NECESSITY IN THEOREM 1.

1. Assume that (2) holds, and let σ_F be the supporting function of F. Let $x_0 \in \partial F \cap W \cap \partial\Omega$. By (2)(b), $\partial F \cap W$ is smooth and therefore one can find a ball $B(z,\rho) \subset F \cap W$ such that

(17) $\partial B(z,\rho) \cap \partial F = \{x_0\}$.

Let us show that

(18) $B_P(z,\rho) \cap W \subset F \cap W$.

Let $\Sigma \subset p^{-1}(0)\setminus 0$ be the closure of the conic set of all exterior normals to ∂F in W. Note that, since W is convex,

(19) $F \cap W = \tilde{F} \cap W$, where $\tilde{F} = \{x \in \mathbf{R}^n : (x, \xi) \leqq \sigma_F(\xi), \xi \in \Sigma\}$.

Since $B(z, \rho) \subset F \subset \tilde{F}$ and $\Sigma \subset p^{-1}(0) \setminus 0$,

$$(z, \xi) + \rho |\xi| \leqq \sigma_{\tilde{F}}(\xi), \qquad \xi \in p^{-1}(0) \setminus 0.$$

Therefore, by (5), $B_P(z, \rho) \subset \tilde{F}$ and (18) follows from (19). By (17), $\partial B_P(z, p) \cap \partial F$ is a face with a common normal $\nu(x_0) = x_0 - z$. Thus, by (6),

$$\partial B_P(z, \rho) \cap \partial F \subset \{x_0 + tp'(\nu(x_0))\}_{t \in \mathbf{R}},$$

and by (2)(a),

(20) $\partial B_P(z, \rho) \cap \partial\Omega \subset \{x_0 + tp'(\nu(x_0))\}_{t \in \mathbf{R}}$.

By (1), for every $\delta > 0$ there is an open set $W_\delta \subset W$, such that *diam* $W_\delta \leqq \delta$ and

(21) $\{x_0 + tp'(\nu(x_0))\}_{t \in \mathbf{R}} \cap \partial\Omega$ is compact in W_δ.

Then (20) and (21) imply that for every $\delta > 0$,

(22) $\partial B_P(z, \rho) \cap \partial\Omega$ is compact in W_δ.

By (2)(a), (18) implies

(23) $\partial B_P(z, \rho) \cap W \subset \overline{\Omega}$, $W \supset W_\delta$,

and by Lemma 2 with δ sufficiently small,

(24) $\partial B_P(Z, P) \cap W_\delta$ is an analytic characteristic hypersurface.

2. Let us quote the following non-uniqueness result (immediate from Theorem 5.2.1, [1]).

LEMMA 3. *Let P be an operator of the real principal type with constant coefficients. Let $\Gamma \subset \mathbf{R}^n$ be an analytic characteristic hypersurface defined in an open set W_0 and $x_0 \in \Gamma$. Then there is a neighbourhood W_1 of x_0 and a function $u \in \mathcal{D}'(W_1)$, such that Γ divides W_1 into two domains, W_+ and W_-, $P(-D)u = 0$ in W and $x_0 \in \operatorname{supp} u \subset \overline{W_-}$.*

Let us apply Lemma 3 to $\Gamma = \partial B_P(z, p)$ in W_δ. The conditions of Lemma 3 are satisfied by (24). Consequently, by Lemma 3 and (23) there is an open set $W_1 \ni x_0$, and a distribution u, such that

(25) $P(-D)u = 0$ in W_1

and

(26) $x_0 \in \operatorname{supp} u \subset \overline{B_P(z,\rho)} \cap W_1 \subset \overline{\Omega}$.

Moreover, by (22) one can restrict W_1 so that

(27) $\varepsilon \equiv \operatorname{dist}(\operatorname{supp} u \cap \partial W_1, \mathbf{R}^n \setminus \Omega) > 0$.

Clearly, there exists $\lambda > 0$ such that for a neighbourhood $W_2 \subset W_1$ of x_0

(28) $\operatorname{dist}(\operatorname{supp} u \cap \partial W_2, \mathbf{R}^n \setminus \Omega) \geqq \lambda$.

Let $\chi \in C_0^\infty(W_1)$ be equal to 1 in W_2 and consider a family of functions

$$u_t = (\chi u)(x - t(z - x_0)), \qquad t > 0.$$

Since $\operatorname{supp} u \subset \overline{B_P(z,\rho)} \cap W_1 \subset \overline{\Omega}$, and $z - x_0$ is an inner normal to $\partial B(z,\rho)$, for $t > 0$ sufficiently small, $u_t \in \mathscr{E}'(\Omega)$,

(29) $\operatorname{dist}(\operatorname{supp} u_t, \mathbf{R}^n \setminus \Omega) \to 0$ as $t \to 0$

and from (28)

(30) $\operatorname{dist}(\operatorname{supp} P(-D)u_t, \mathbf{R}^n \setminus \Omega) \geqq \lambda/2$.

By (29), (30) Ω is not P-convex and the proof of necessity is completed.

3. Sufficiency in Theorem 1

Assume that Ω is not P-convex. By Theorem 10.6.3, [2], there is $u \in \mathscr{E}'(\Omega)$ such that

(31) $r_1 \equiv \operatorname{dist}(\operatorname{supp} P(-D)u, \mathbf{R}^n \setminus \Omega) > r \equiv \operatorname{dist}(\operatorname{supp} u, \mathbf{R}^n \setminus \Omega)$.

Let $K \subset \mathbf{R}^n$ be a compact convex set with a supporting function $\sigma > 0$ and let

(32) $F_\sigma = \{x \in \mathbf{R}^n : \exists\, y \in \operatorname{supp} u,\ x - y \in K\}$.

One may write (32) also as

(33) $x \in F_\sigma \Leftrightarrow \min_{y \in \operatorname{supp} u} \max_{\xi \in \mathbf{R}^n \setminus 0} ((x - y, \xi)/\sigma(\xi)) < 1$.

Let

(34) $\sigma_0(\xi) = r\,|\xi|$.

Note that by (31)

(35) $F_{\sigma_0} \subset \overline{\Omega}$, $\partial F_{\sigma_0} \cap \partial\Omega \neq \varnothing$.

LEMMA 4. *Let* $x_0 \in \partial\Omega \cap F_{\sigma_0}$. *Then for every* $x \in B(x_0, r_1 - r)$

(36) $\operatorname{dist}(x, \operatorname{supp} u) = \inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \sup_{\xi \in p^{-1}(0) \setminus 0} ((x-y, \xi)/|\xi|)$.

PROOF. For $x \in B(x_0, r_1 - r)$,

$$\rho \equiv \operatorname{dist}(x, \operatorname{supp} u) = \min_{y \in \operatorname{supp} u} \max_{\xi \in \mathbf{R}^n \setminus 0} \frac{(x-y, \xi)}{|\xi|} .$$

Assume that the value ρ is attained at some $y_0 \in \operatorname{supp} u$. Then

$$|x_0 - y_0| \leqq |x - x_0| + |x - y_0| < r_1,$$

and therefore

(37) $\rho = \inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \max_{\xi \in \mathbf{R}^n \setminus 0}((x-y, \xi)/|\xi|)$.

By Lemma 1, $u = 0$ in $B_P(x, \rho) \cap B(x_0, r_1)$, i.e., if

(38) $\max_{\xi \in p^{-1}(0) \setminus 0} ((x-y, \xi)/|\xi|) < \rho$ and $y \in B(x_0, r_1)$,

then $y \notin \operatorname{supp} u$. Let

(39) $\rho' = \inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \max_{\xi \in p^{-1}(0) \setminus 0} ((x-y, \xi)/|\xi|)$.

By (37), (39),

(40) $\rho' \leqq \rho$.

If $\rho' = \rho$, the lemma is proved. Assume that

(41) $\rho' < \rho$.

Then there exists $y_0 \in \operatorname{supp} u \cap B(x_0, r_1)$ such that (38) holds and therefore $y_0 \notin \operatorname{supp} u$. Thus by contradiction (41) is not valid, and the proof is completed.

LEMMA 5. *Let* $x_0 \in \partial F_{\sigma_0} \cap \partial\Omega$, $\sigma_0(\xi) = r|\xi|$ *and* $r < r_2 < r_1$. *For every convex homogeneous function* σ *such that*

(42) $\sigma(\xi) = \sigma_0(\xi)$ *for* $\xi \in p^{-1}(0)$, $\sigma_0(\xi) \leqq \sigma(\xi) \leqq r_2|\xi|$, $\sigma(-\xi) = \sigma(\xi)$

the relation

(43) $F_{\sigma_0} \cap B(x_0, r_1 - r_2) = F_\sigma \cap B(x_0, r_1 - r_2)$.

holds.

PROOF. If $x \in F_\sigma \cap B(x_0, r_1 - r_2)$, then by (33), (42),

$$\min_{y \in \operatorname{supp} u} \max_{\xi \in \mathbf{R}^n \setminus 0} \frac{(x - y, \xi)}{\sigma(\xi)} \leqq 1.$$

Let the minimum be attained at $y_0 \in \operatorname{supp} u$. Then since $\sigma(\xi) \leqq r_2 |\xi|$ by (42),

$$|x_0 - y_0| \leqq |x - y_0| + |x - x_0| < r_2 \max_{\xi \in \mathbf{R}^n \setminus 0} \frac{(x - y_0, \xi)}{\sigma(\xi)} + r_1 - r_2 < r_1$$

and $y_0 \in B(x_0, r_1)$. By (42),

$$\inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \max_{\xi \in p^{-1}(0) \setminus 0} \frac{(x - y, \xi)}{\sigma_0(\xi)} = \inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \max_{\xi \in p^{-1}(0) \setminus 0} \frac{(x - y, \xi)}{\sigma(\xi)}$$

$$(44) \qquad \leqq \inf_{y \in \operatorname{supp} u \cap B(x_0, r_1)} \max_{\xi \in \mathbf{R}^n \setminus 0} \frac{(x - y, \xi)}{\sigma(\xi)} \leqq 1$$

and by Lemma 4, $x \in F_{\sigma_0} \cap B(x_0, r_1 - r_2)$. Conversely, since $\sigma_0(\xi) \leqq \sigma(\xi)$, $F_{\sigma_0} \subset F_\sigma$ and the lemma is proved.

We can now complete the proof of Theorem 1.

1. Let us take $x_0 \in \partial F_{\sigma_0} \cap \partial \Omega$. By the definition of F_{σ_0} (see (32), (34)) there exists $y_0 \in \operatorname{supp} u$ such that

(45) $|x_0 - y_0| = r$.

Consider the compact convex set

(46) $K = \overline{B_P(y_0, r) \cap B(x_0, r_2)}$ for $r < r_2 < r_1$

and let σ be the supporting function of K. Clearly, σ satisfies (42) and therefore, by Lemma 5, $K \cap B(x_0, r_1 - r_2) \subset F_{\sigma_0}$ and by (35)

(47) $K \cap B(x_0, r_1 - r_2) \cap \overline{\Omega}$.

2. By (31), (45), $y_0 \in \operatorname{supp} u \setminus \operatorname{supp} P(-D)u$. Since $B(x_0, r) \cap \operatorname{supp} u = \varnothing$ and $y_0 \in \partial B(x_0, r)$, the Holmgren uniqueness theorem implies that the normal to $\partial B(x_0, r)$ at y_0 is characteristic, i.e.,

(48) $x_0 - y_0 \in p^{-1}(0)$.

Therefore, by (5), $x_0 \in \partial K$. Moreover, since $x_0 \in \partial B(y_0, r)$, $x_0 - y_0$ is also the exterior normal to $\partial B(y_0, r)$ at x_0. By Lemma 2, the face of K with the common exterior normal $x_0 - y_0$ is an interval on $t \to x_0 + tp'(x_0 - y_0)$.

3. Note that since $x_0 - y_0$ is an exterior normal to $\partial B(y_0, r)$ at x_0,

(49) $B(y_0 + (1 - r_0/r)(x_0 - y_0), r_0) \subset B(y_0, r)$ for $0 < r_0 < r$

and the only common point of the boundaries of these two sets is x_0 where the common exterior normal is $x_0 - y_0$.

Therefore, by (5),

(50) $B_P(y_0 + (1 - r_0/r)(x_0 - y_0), r_0) \subset B_P(y_0, r)$,

and, due to Lemma 2,

(51) $x_0 \in \partial B_P(y_0 + (1 - r_0/r)(x_0 - y_0), r_0)$
$\cap\, \partial B_P(y_0, r) \subset \{x_0 + tp'(x_0 - y_0), t \in \mathbf{R}\}$.

4. Consider the set

(52) $F = \overline{B_P(y_0 + (1 - r_0/r)(x_0 - y_0), r_0) \cap B(x_0, r_1 - r_2)}$.

By (50), (46)

(53) $F \subset K$

and by (47), (53), F satisfies (2)(a). By Lemma 2 and (52) there is an open convex set $W_0 \ni x_0$, $W_0 \subset B(x_0, r_1 - r_2)$, such that $\partial F \cap W_0$ is an analytic hypersurface with normals in $p^{-1}(0)$. Thus, F satisfies (2)(b) for any convex $W \subset W_0$.

By (50), (51)

$$x_0 \in \partial F \cap \partial K \subset \{x_0 + tp'(x_0 - y_0), t \in \mathbf{R}\}.$$

Therefore by (47)

(54) $x_0 \in \partial F \cap \partial \Omega \subset \{x_0 + tp'(x_0 - y_0), t \in \mathbf{R}\}$.

Then by (1) there is an open convex set $W \subset W_0$ such that

(55) $\partial F \cap \partial \Omega$ is compact in W.

By (54),

(56) $\partial F \cap \partial \Omega = \varnothing$

and by (55), (56), F satisfies (2)(c).

Theorem 1 is proved.

4. P-convexity and uniqueness theorems. Example of a P-convex set

1. Assume for the sake of simplicity that Ω is bounded and $\partial\Omega$ is analytic and satisfies a certain geometric condition (A). We will say that (A) is a local uniqueness condition at $x_0 \in \partial\Omega$ for the operator P if from (A) it follows that for every neighbourhood V of x_0 there exists a neighbourhood W of x_0 such that $u \in \mathscr{D}'(\mathbf{R}^n)$, $Pu = 0$ in V and $u = 0$ in $V/\overline{\Omega}$ imply $u = 0$ in W. For example, if $\xi(x_0)$ is the normal to $\partial\Omega$ at x_0, then

(57) $p(\xi(x_0)) = 0$

is a local uniqueness condition at x_0 (by the Holmgren uniqueness theorem). Other uniqueness conditions are also known:

(58) there exists a sequence $t_j \to 0$ such that $x_0 + t_j p'(\xi(x)) \notin \overline{\Omega}$

and

(59) the twisted surface condition; see Definition 3.1, [5].

For more details see Sections 1–3 in [5] and references cited therein.

If at every point of $\partial\Omega$ either (57), or (58) or (59) holds, then by local uniqueness and since $\partial\Omega$ is compact, $\operatorname{dist}(\operatorname{supp} Pu, \mathbf{R}^n \setminus \Omega) \geqq \varepsilon > 0$ and $\operatorname{supp} u \in \overline{\Omega}$ imply that $\operatorname{dist}(\operatorname{supp} u, \mathbf{R}^n \setminus \Omega)$ has a uniform positive lower bound, and therefore Ω is $P(-D)$ convex. In the paper [3] a uniqueness condition, similar to (59), was established to study P-convexity and it is the weakest uniqueness condition in $\mathbf{R}^3$ found in the literature.

2. We wish to present an example of a P-convex set in $\mathbf{R}^4$ when

(60) $p(\xi) = \xi_1\xi_3 + \xi_2^2 - \xi_4^2$.

Let

(61) $f(x) = x_3 + x_1|x_2| - x_1^4 + x_3x_4$

and

(62) $\Omega_\varepsilon = \{x \in \mathbf{R}^4 : |x| < \varepsilon, f(x) < 0\}$, $0 < \varepsilon < 1$.

We wish to show that Ω_ε is P-convex when ε is small. Although Ω_ε does not satisfy the condition (1) of Theorem 1, Theorem 1 is still applicable for the following reason: The actual condition used in our proof is weaker than (1):

(1′) If F is a closed convex set and W is an open convex set such that $F \subset \overline{\Omega}$ and $\partial F \cap W$ is an analytic hypersurface with normals $\xi(x) \in p^{-1}(0)$, then for every $x \in \partial F \cap \partial\Omega \cap W$, the set $\{x + tp'(\xi(x))\}_{t\in\mathbf{R}} \cap \partial\Omega$ does not contain any interval of $\{x + tp'(\xi(x))\}_{t\in\mathbf{R}}$.

Clearly, if (1) holds, then $\partial\Omega$ does not contain intervals of straight lines, in particular of those specified in (1′).

We will therefore apply Theorem 1 with (1′) replacing (1) to show that Ω_ε is P-convex when ε is small. Some details of the proof will be left to the reader.

3. Assume that Ω_ε is not P-convex. Then there exist sets F, W satisfying (2). Let $x \in \partial F \cap \partial\Omega_\varepsilon \cap W$. Then by (2a), (2b), $\partial\Omega_\varepsilon$ has a characteristic interior normal $\tilde{\xi} \in p^{-1}(0)$. Moreover, since $\tilde{x} + tp'(\tilde{\xi}) \in \partial F$ when t is small, (2a) implies $\underline{\tilde{x} + tp'(\tilde{\xi}) \in \bar{\Omega}_\varepsilon}$ for t small. Therefore $\tilde{x}$ cannot lie on the spherical portion $\partial B_\varepsilon \cap \partial\Omega_\varepsilon$ of $\partial\Omega_\varepsilon$: either such an $\tilde{x}$ would not be characteristic or the line $t \to \tilde{x} + tp'(\tilde{\xi})$ would be tangent to ∂B_ε and lie in the exterior. The remaining part of $\partial\Omega_\varepsilon$ is smooth when $x_2 \neq 0$ and it is easy to compute the value of p on the normal vector:

$$(63)\quad p(f'(x))\,\big|_{f(x)=0} = (1 + x_4)(|x_2| - 4x_1^3) + x_1^2 - \left(\frac{x_1|x_2| - x_1^4}{1 + x_4}\right)^2.$$

One can observe that if ε is small, none of $x \in \{x \in \mathbf{R}^4 : (x) = 0,\ x_2 \neq 0,\ |x| < \varepsilon\}$ is characteristic and, necessarily, $\tilde{x}_2 = 0$. Consider the case $\tilde{x}_2 = 0$, $\tilde{x}_1 < 0$. At such points, $\partial\Omega_\varepsilon \cap \bar{B}_\varepsilon$ has a cone of interior normals

$$(64)\quad \tilde{\xi}(\alpha) = (-4\tilde{x}_1^3, \alpha\tilde{x}_1, 1 + \tilde{x}_4, \tilde{x}_3),\ |\alpha| \leqq 1.$$

Computing $p(\tilde{\xi}(\alpha))$ one can see that it is positive when ε is small. Consider the case $\tilde{x}_2 = 0$, $\tilde{x}_1 > 0$. At these points $\partial\Omega_\varepsilon \cap \tilde{B}_\varepsilon$ has a cone of exterior normals and therefore $\tilde{x} \notin \partial\Omega_\varepsilon \cap F$ since $\partial F \subset \bar{\Omega}_\varepsilon$ is a smooth surface.

Thus, necessarily, $\tilde{x}_1 = \tilde{x}_2 = 0$. Since $\tilde{x} \in f^{-1}(0)$, by (61), $\tilde{x}_3 = 0$ and $\tilde{x} = (0, 0, 0, \tilde{x}_4)$. This is a characteristic point with $\tilde{\xi} = (0, 0, 1 + x_4, 0)$ and $p'(\tilde{\xi}) = (1 + x_4, 0, 0, 0)$. Note that $\tilde{\xi} \neq 0$ since $|x| < \varepsilon < 1$. Remarkably, this is not a uniqueness point by condition (58), and condition (59) is formulated only for smooth surfaces (although it is perhaps possible to apply here the technique of [5], [6], too).

Now, since ∂F is a smooth characteristic surface, it contains a biocharacteristic line $t \to \tilde{x} + tp'(\tilde{\xi})$ and at every point of this line the normal to ∂F is parallel to $\tilde{\xi}$, and, consequently, the surface

$$S_N = \{x_3 = -N(x_2^2 + (x_4 - \tilde{x}_4)^2)\} \qquad \text{with } N > 0 \text{ sufficiently large}$$

lies in F. Let us compute the value of f at $x \in S_N$:

$$(65)\quad f(x) = x_1|x_2| - x_1^4 - N(1 + x_4)(x_2^2 + (x_4 - \tilde{x}_4)^2).$$

Set $x_4 = \tilde{x}_4$, $M = (1 + \tilde{x}_4)N$, $x_2 = \xi$, $x_1 = (M + 1)s$. Then

$$(66)\quad f(x) = s^2 - (M + 1)^4 s^4.$$

Since $\varepsilon < 1$ and $|\tilde{x}_4| < |\tilde{x}| < \varepsilon$, the number M is positive, and for every M (resp. N) there is a neighbourhood of $s = 0$ where $f(x) > 0$. Thus $S_N \setminus \Omega_\varepsilon \neq \varnothing$ and, since $S_N \subset F$, the set F does not satisfy (2). We conclude therefore that for ε sufficiently small, Ω_ε is P-convex.

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